Exponential Stabilization of Nonlinear Systems by an Estimated State Feedback

Abderrahman Iggidr, Gauthier Sallet

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A. Iggidr G.Sallet
CONGE Project, INRIA Lorraine & University of Metz
4, rue Marconi 57 070 METZ –FRANCE
e.mail : {iggidr,sallet} @ilm.loria.fr

Abstract: In this paper we investigate the stabilizability problem of a class of multi-input multi- output nonlinear systems which linearization at the origin is controllable and observable. Under assumptions on the nonlinear part we prove : (a) the system is globally exponentially stabilizable (G.E.S) by means of linear feedback law. (b) the system can be G.E.S using a state estimation given by an observer.

Keywords: exponential stabilization, feedback, nonlinear systems, Lyapunov functions, observer.

1 Introduction

We consider a nonlinear system of the form :

$$\begin{cases} \dot{x} = Ax + Bu + g(x,u) \\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^r,\ A \in M_{n,n}(\mathbb{R}),\ B \in M_{n,r}(\mathbb{R}) \end{cases} \tag{1}$$

where $M_{n,m}(\mathbb{R})$ is the set of matrices with n rows and m columns and the map

$$f = (f_1, .., f_n)^T : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

is Lipschitz continuous such that $f(0,0) = 0$. We assume that the pair (A, B) is controllable and is in Brunovsky canonical form, i.e.

- A is a block–diagonal matrix of the form

$$A = \begin{pmatrix} A_{k_1} & 0 & . & . & . & 0 \\ 0 & . & . & & & . \\ . & . & . & . & & . \\ . & & . & . & . & . \\ . & & & . & . & 0 \\ 0 & . & . & . & 0 & A_{k_r} \end{pmatrix}$$

where A_{k_i}, $1 \le i \le r$, is a matrix in $M_{k_i,k_i}(\mathbb{R})$ given by

$$A_{k_i} = \begin{pmatrix} 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & & . \\ . & & . & . & . & . \\ . & & & . & . & 0 \\ 0 & . & . & . & 0 & 1 \\ 0 & . & . & . & . & 0 \end{pmatrix}$$

- B is a block–diagonal matrix of the form

$$B = \begin{pmatrix} b_{k_1} & 0 & . & . & . & 0 \\ 0 & . & . & & & . \\ . & . & . & . & & . \\ . & & . & . & . & . \\ . & & & . & . & 0 \\ 0 & . & . & . & 0 & b_{k_r} \end{pmatrix}$$

where b_{k_i}, $1 \le i \le r$, is a column–vector in $\mathbb{R}^{k_i}$ given by

$$b_{k_i} = \begin{pmatrix} 0 \\ . \\ . \\ 0 \\ 1 \end{pmatrix}$$

In this paper, we study nonlinear systems of the form (1) which have the following property :

(H1) *There exists a positive constant K such that for any $i = 1, \ldots, n$ the following holds :*

$$\begin{cases} |g_i(x,u)| \le K \,\|(x_1, \ldots, x_i, 0, \ldots, 0)\| \\ \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n\ ,\ \forall u \in \mathbb{R}^r \end{cases} \tag{2}$$

where $\|\quad\|$ is the usual Euclidean norm on $\mathbb{R}^n$.

In section 2, we shall see that if condition (**H1**) is satisfied , then system (1) is globaly exponentially stabilizable

(G.E.S.) at the origin by means of a linear feedback. This is a generalization of a result of Tsinias [3] who studied the single input systems, our proof is different from its one and it is based on an idea from [1].

In section 3, we suppose in addition
(H2) $g_i(x,u) = g_i(x_1, \ldots, x_i, 0, \ldots, 0, u)$
so we can construct an observer for system (1) with the output

$$y = Cx \tag{3}$$

where

$$C = \begin{pmatrix} C_{k_1} & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & C_{k_p} \end{pmatrix}$$

$$C_{k_i} = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}$$

This observer is of the form :

$$\dot{\hat{x}} = A\hat{x} + Bu + g(\hat{x}, u) - \tilde{E}(C\hat{x} - y) \tag{4}$$

We show that the error will tend to zero with an exponential rate of convergence and we construct a dynamic feedback which globally exponentially stabilizes system (1-3).

2 Stabilization

Before given our main result we recall some basic notions about exponential stability. Let us consider a system of ordinary differential equations

$$\dot{x} = X(x) \;, \quad x \in \mathbb{R}^n \tag{5}$$

We supoose that the origin is an equilibrium point for the vector field X i.e, $X(0) = 0$ and we denote $X_t(x)$ the solution of (5) starting from the point x at $t = 0$ ($X_0(x) = x$). We say that (5) is *globally exponentially stable* at the origin if there exist positive constants M and α such that

$$\|X_t(x)\| < M\|x\|e^{-\alpha t}$$

for any initial condition x in $\mathbb{R}^n$ and any $t > 0$. To prove exponential stability we shall use the following Lyapunov theorem [2] :

Theorem *The solution $x_t \equiv 0$ of the equation (5) is globally exponentially stable if there exist a Lyapunov function V and three positive constants k_1, k_2, k_3 such that for any $x \in \mathbb{R}^n$ one has :*

$$k_1\|x\|^2 \leq V(x) \leq k_2\|x\|^2$$

and

$$\dot{V}(x) = X.V(x) = \langle \nabla V(x), X(x) \rangle \leq -k_3\|x\|^2$$

We shall say that a control system

$$\begin{cases} \dot{x} = f(x,u) \;,\; x \in \mathbb{R}^n \;,\; u \in \mathbb{R}^r \\ f(0,0) = 0 \end{cases}$$

is globally exponentially stabilizable (G.E.S) at the origin of $\mathbb{R}^n$ if there exists a continuous feedback

$$\begin{array}{rccc} u \;: & \mathbb{R}^n & \rightarrow & \mathbb{R}^r \\ & x & \mapsto & u(x) \end{array}$$

such that the closed loop system $\dot{x} = f(x, u(x))$ is globally exponentially stable at the origin.

The aim of this section is to prove that system (1) is G.E.S and to give explicitly the stabilizing feedback provided that assumption **(H1)** is satisfied. To this end let $\alpha \in \mathbb{R}$, $\alpha > 1$ and introduce the following matrix :

$$\Phi = \begin{pmatrix} \alpha^{-1} & 0 & \ldots & 0 \\ 0 & \alpha^{-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \alpha^{-n} \end{pmatrix}$$

One can write

$$\Phi = \begin{pmatrix} \Phi_{k_1} & 0 & \ldots & 0 \\ 0 & \Phi_{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \Phi_{k_r} \end{pmatrix}$$

where r is the number of the blocs of matrix A and

$$\Phi_{k_i} = \begin{pmatrix} \alpha^{-(k_{i-1}+1)} & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \alpha^{-(k_{i-1}+k_i)} \end{pmatrix}$$

with $k_0 = 0$

Using the above decomposition, a simple computation can prove the following :

lemma 1 *Matrices A, B and Φ defined above satisfy :*

i) $\alpha\, \Phi^{-1} A\, \Phi = A$

ii) $\forall F \in M_{r,n}(\mathbb{R})$ *there exists* $\tilde{F} \in M_{r,n}(\mathbb{R})$ *such that*

$$B\tilde{F} = \alpha\, \Phi^{-1} BF\, \Phi$$

and $\tilde{F}$ is given by the following formula :

$$\tilde{F} = \alpha\, B^T\, \Phi^{-1} BF\, \Phi$$

iii) $\forall x \in \mathbb{R}^n \;: \quad \alpha^{-n}\, \|x\| \leq \|\Phi x\| \leq \alpha^{-1}\, \|x\|$

We can now prove the main result of this section.

Theorem 1 *If the assumption* (**H1**) *holds then the system (1) is G.E.S. at the origin by means of a linear feedback*

$$u = \tilde{F}\, x$$

where $\tilde{F}$ is defined by (ii) and F is such that $(A + BF)$ has all its eigenvalues with negative real part.

Proof
Since the pair (A, B) is controllable there exists a matrix $F \in M_{r,n}(\mathbb{R})$ such that $(A + BF)$ has all its eigenvalues with negative real part. Let $M = A + BF$, there exists a symmetric positive definite matrix S such that $M^T S + SM = -Q$, Q symmetric positive definite .

Now consider the function :

$$V(x) = x^T\, \Phi\, S\, \Phi\, x$$

V is positive definite and proper (V is a quadratic Lyapunov function). Let us evaluate its derivative along the trajectories of the closed-loop system

$$\dot{x} = (A + B\tilde{F})x + g(x, \tilde{F}x) = \tilde{M}x + g(x, \tilde{F}x) \quad (6)$$

$\dot{V}(x) = x^T(\, \Phi\, S\, \Phi\, \tilde{M} + \tilde{M}^T\, \Phi\, S\, \Phi\,)x + 2x^T\, \Phi\, S\, \Phi\, g(x, \tilde{F}\,x)$ where $\tilde{M} = A + B\tilde{F}$

Taking into account (i) and (ii) we have

$$\tilde{M} = A + B\tilde{F} = \alpha\Phi^{-1}A\Phi + \alpha\Phi^{-1}BF\Phi$$

$$\tilde{M} = \alpha\Phi^{-1}(A + BF)\Phi = \alpha\Phi^{-1}M\Phi$$

Therefore :

$$\begin{aligned} \dot{V}(x) = {} & \alpha\, x^T(\Phi\, S\, M\, \Phi + \Phi\, M^T\, S\, \Phi)\, x \\ & +2\, x^T\Phi\, S\, \Phi\, g(x, \tilde{F}x) \\ \dot{V}(x) = {} & -\alpha\, x^T\Phi\, Q\, \Phi\, x + 2\, x^T\Phi\, S\, \Phi\, g(x, \tilde{F}x) \end{aligned}$$

The matrix Q is symmetric definite positive so there exists a positive constant a such that :

$$x^T\Phi Q\Phi x \geq a \parallel \Phi x \parallel^2$$

where

$$a = inf\left\{z^T Q z \;/\; z \in S^{n-1} \text{ the unit sphere in } \mathbb{R}^n\right\}$$

And then :

$$\dot{V}(x) \leq -\alpha\; a\; \|\Phi x\|^2 + 2\,\|\Phi x\|\;\; \|S\,\|\;\; \left\|\Phi g(x, \tilde{F}x)\right\|$$

According to (**H1**) we have

$$\begin{aligned} & \left\|\Phi g(x, \tilde{F}x)\right\|^2 = \sum_{i=1}^{n} \frac{1}{\alpha^{2i}} g_i^2(x, \tilde{F}x) \\ & \leq K^2 \sum_{i=1}^{n} \frac{1}{\alpha^{2i}}(x_1^2 + \cdots + x_i^2) \\ & \leq K^2 \sum_{j=1}^{n} \left(\frac{x_j}{\alpha^j}\right)^2 (1 + \frac{1}{\alpha^2} + \cdots + \frac{1}{\alpha^{2(n-j)}}) \\ & \leq nK^2\, \|\Phi x\|^2 \end{aligned}$$

So

$$\dot{V}(x) \leq (-\alpha\, a + 2\sqrt{n}\, K\, \|S\|)\; \|\Phi x\|^2$$

If we choose $\alpha > \; Max\left(1\;,\; \dfrac{2\sqrt{n}\,K\,\|S\|}{a}\right)$ then

$$\dot{V} \leq -c\,\|\Phi x\|^2$$

where c is a positive constant.

Since $\|\Phi x\| \geq \dfrac{1}{\alpha^n}\,\|x\|$ it follows that :

$$\dot{V}(x) \leq -c'\,\|x\|^2$$

where c' is a positive constant , and this completes the proof of theorem 1. ■

Example 1. The three dimensional system

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 + x_1 cos(u(x_2^2 + x_1 + x_3)) \\ \dot{x}_2 = x_3 + \dfrac{x_1 + x_2}{1 + x_3^2} \\ \dot{x}_3 = \sqrt{x_1^2 + x_2^2 + x_3^2}\, e^{-u^2} + u \end{array} \right. \quad (7)$$

has the form (1) with

$$g(x, u) = \begin{pmatrix} x_1 cos(u(x_2^2 + x_1 + x_3)) \\ \dfrac{x_1 + x_2}{1 + x_3^2} \\ \sqrt{x_1^2 + x_2^2 + x_3^2}\, e^{-u^2} \end{pmatrix}$$

which satisfies condition (**H1**), so system (7) is G.E.S and s stabilizing feedback can be computed according to theorem 1 as follows : We choose $F = (-1, -3, -3)$, $Q = -Id_{\mathbb{R}^3}$ and we solve $M^T S + SM = Q$:

$$S = \begin{pmatrix} \frac{37}{16} & \frac{31}{16} & \frac{1}{2} \\ \frac{31}{16} & \frac{13}{4} & \frac{13}{16} \\ \frac{1}{2} & \frac{13}{16} & \frac{7}{16} \end{pmatrix}$$

We compute $\tilde{F} = \alpha\; B^T\; \Phi^{-1} BF\; \Phi$. The feedback law

$$u = -\alpha^3 x_1 - 3\alpha^2 x_2 - 3\alpha x_3$$

globally exponentially stabilizes system (7) if we choose

$$\alpha > \; Max\left(1\;,\; \frac{2\sqrt{n}\,K\,\|S\|}{a}\right) \simeq 10\sqrt{3}$$

.

3 Construction of the observer and stabilization using a state estimation

Theorem 2 *If the condition* (**H1**) *and* (**H2**) *are satisfied then there exists* $\tilde{E} \in M_{n,p}(\mathbb{R})$ *such that the system (4) is an exponential observer for (1-3).*

Proof
The error $e = \hat{x} - x$ satisfies the following equation :

$$\dot{e} = (A - \tilde{E}C)e + g(\hat{x}, u) - g(x, u) \tag{8}$$

since (A,C) is observable, there exists $E \in M_{n,p}(\mathbb{R})$ such that $(A - EC)$ has all its eigenvalues with negative real part.
Remark that :

$$|g_i(x,u) - g_i(\hat{x},u)| \leq K \, \|p_i(x - \hat{x})\| = K \, \|(e_1, .., e_i)\|$$

where $p_i : \mathbb{R}^n \to \mathbb{R}^i$ is the canonical projection.
So if we take

$$\tilde{E} = \alpha \, \Phi^{-1} EC \, \Phi C^T$$

then , according to the proof of theorem 1, there exists a quadratic Lyapunov function W such that for α large enough :

$$\dot{W}(e) \leq -b \, \|e\|^2 \, , b > 0$$

This proves that $\|e(t)\| \leq M \|e_0\| e^{-\alpha t}$ so (4) is an exponential observer for (1-3). ■

Now we use the above results to achieve the stabilization of system (1) with the state estimation given by the observer (4). Consider the following system defined on $\mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{cases} \dot{x} = Ax + B\tilde{F}\hat{x} + g(x, \tilde{F}\,\hat{x}) \\ \dot{e} = (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \end{cases} \tag{9}$$

Theorem 3 *If the assumption* (**H1**) *and* (**H2**) *hold then (9) is globally exponentially stable i.e. the closed- loop system (1-3), with the state estimation given by the observer (4) is G.E.S.*

Proof
Since $e = \hat{x} - x$, system (9) becomes :

$$\begin{cases} \dot{x} = (A + B\tilde{F})x + B\tilde{F}e + g(x, \tilde{F}\hat{x}) \\ \dot{e} = (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \end{cases} \tag{10}$$

According to the proofs of theorem 1 and theorem 2, there exist two quadratic Lyapunov functions V and W such that :

$$\langle \nabla V(x), (A + B\tilde{F})x + g(x, \tilde{F}\hat{x}) \rangle \quad \leq \; -c' \; \|x\|^2$$

$$\langle \nabla W(e), (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \rangle \quad \leq \; -b \, \|e\|^2$$

where $\langle ., . \rangle$ is the usual Euclidean inner product on $\mathbb{R}^n$.

Let U be the function defined on $\mathbb{R}^n \times \mathbb{R}^n$ by :

$$U(x\,, e) = \beta \, V(x) + W(e) \; , \; \beta > 0$$

U is a quadratic positive definite function and we have :

$$\begin{aligned} \dot{U}(x, e) = \; & \beta \langle \nabla V(x), (A + B\tilde{F})x + g(x, \tilde{F}\hat{x}) \rangle \\ & + \beta \langle \nabla V(x), B\tilde{F}e \rangle \\ & + \langle \nabla W(e), (A - \tilde{E}C)e + g(\hat{x}, \tilde{F}\hat{x}) - g(x, \tilde{F}\hat{x}) \rangle \end{aligned}$$

So

$$\dot{U}(x\,, e) \leq -\beta \, c' \; \|x\|^2 + 2 \, \beta \, K' \, \|x\| \; \|e\| - b \; \|e\|^2$$

where K' is a positive constant defined by :

$$\langle \, \nabla \, V(x) \; , \; B \, \tilde{F} \, e \, \rangle \leq 2 \; K' \, \|x\| \; \|e\|$$

If we choose β such that $0 < \beta < \dfrac{c' \; b}{K'^2}$ then $\dot{U}(x\,, e)$ is negative definite on $\mathbb{R}^n \times \mathbb{R}^n$ and so theorem 3 is proved.

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